Homework exercise

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1 Main section

The main problem: given $h_k > 0, k = 0, \dots, N$ prove that

$$\Delta_N = \frac{R^2 + \sum\limits_{i=0}^{N} h_i^2}{2 \sum\limits_{i=0}^{N} h_i}$$

is a convex function of h_k.

In this proof we will use the following inequalities:

Theorem 1 (Cauchy inequality)**.** *For positive a and b it holds $\frac{a+b}{2} \geq \sqrt{ab}$.*

Theorem 2 (Cauchy-Bunyakovsky-Schwarz inequality)**.** *Let $x_1, x_2 \in \mathbb{R}^n$. Then $(x_1, x_2) \leq ||x_1|| \cdot ||x_2||$.*

Let's prove the following lemma.

Lemma 1. *Let $x_1, x_2 \in \mathbb{R}^n_+$, i.e. all coordinates are positive. Let 1 denote a vector of all ones. Then it holds*

$$2(x_1, x_2) \leq \frac{||x_1||^2}{(x_1, 1)}(x_2, 1) + \frac{||x_2||^2}{(x_2, 1)}(x_1, 1)$$

Proof. By Cauchy inequality (noticing everything is positive)

$$\frac{\frac{||x_1||^2}{(x_1,1)}(x_2, 1) + \frac{||x_2||^2}{(x_2,1)}(x_1, 1)}{2} \geq \sqrt{\frac{||x_1||^2}{(x_1, 1)}(x_2, 1)\frac{||x_2||^2}{(x_2, 1)}(x_1, 1)} = ||x_1|| \cdot ||x_2||$$

Then by Cauchy-Bunyakovsky-Schwarz inequality $||x_1|| \cdot ||x_2|| \geq (x_1, x_2)$. QED $\square$

The next lemma will prove the hard part of the function we want to show convexity later.

Lemma 2. *Let $x \in \mathbb{R}^n_+$. Then the function $f(x) = \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i}$ is convex.*

Proof. Firstly, notice that $f(x) = \frac{||x||^2}{(x,1)}$.

Take $x_1, x_2 \in \mathbb{R}^n_+$. Fix any $\lambda \in [0;1]$. We prove convexity by definition. For the only inequality we apply the lemma above.

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &= \frac{||\lambda x_1 + (1-\lambda)x_2||^2}{(\lambda x_1 + (1-\lambda)x_2, 1)} = \frac{\lambda^2||x_1||^2 + 2\lambda(1-\lambda)(x_1,x_2) + (1-\lambda)^2||x_2||^2}{\lambda(x_1,1) + (1-\lambda)(x_2,1)} \leq \\ &\leq \frac{\lambda^2||x_1||^2 + \lambda(1-\lambda)\left(\frac{||x_1||^2}{(x_1,1)}(x_2,1) + \frac{||x_2||^2}{(x_2,1)}(x_1,1)\right) + (1-\lambda)^2||x_2||^2}{\lambda(x_1,1) + (1-\lambda)(x_2,1)} = \\ &= \frac{\lambda||x_1||^2\left(\lambda + (1-\lambda)\frac{(x_2,1)}{(x_1,1)}\right) + (1-\lambda)||x_2||^2\left((1-\lambda) + \lambda\frac{(x_1,1)}{(x_2,1)}\right)}{\lambda(x_1,1) + (1-\lambda)(x_2,1)} = \\ &= \frac{\lambda||x_1||^2\left(\frac{\lambda(x_1,1)+(1-\lambda)(x_2,1)}{(x_1,1)}\right) + (1-\lambda)||x_2||^2\left(\frac{\lambda(x_1,1)+(1-\lambda)(x_2,1)}{(x_2,1)}\right)}{\lambda(x_1,1) + (1-\lambda)(x_2,1)} = \\ &= \lambda\frac{||x_1||^2}{(x_1,1)} + (1-\lambda)\frac{||x_2||^2}{(x_2,1)} = \lambda f(x_1) + (1-\lambda)f(x_2) \end{aligned}$$

□

Now basically we are almost done. Here is an easy proof of the next auxiliary function.

Lemma 3. *Let $x \in \mathbb{R}^n_+$. Then the function $g(x) = \frac{1}{\sum_{i=1}^{n} x_i}$ is convex.*

Proof. Every partial derivative of g is $\frac{\partial^2 g}{\partial x_i \partial x_j} = \frac{2}{\left(\sum_{i=1}^{n} x_i\right)^3}$. Then the hessian of g is $H_g = \frac{2}{\left(\sum_{i=1}^{n} x_i\right)^3} J_n$ where J_n is the matrix of all ones. From J_n is positive semidefinite, we immediately conclude that H_g is positive semidefinite and $g(x)$ is convex on $\mathbb{R}^n_+$. □

And the final result.

Theorem 3. *Δ_N is convex.*

Proof. Rewrite $\Delta_N = \frac{R^2}{2}g(h) + \frac{1}{2}f(h)$. Sum of convex functions multiplied by positive scalars in convex. □